

A Compendium of Light-Cone Quantization

Hans-Christian Pauli

*Max-Planck-Institut für Kernphysik, D-69029 Heidelberg, Germany.
pauli@mpi-hd.mpg.de*

Abstract

For the purpose of consistent notation and easy reference the most important relations in light-cone quantization are compiled from a recent review [1], where all further details and derivations can be found.

In the wake of the success of Feynman's action-oriented approach, the simultaneous work by Dirac [2] on the Hamiltonian approach in covariant theories has been forgotten for a long time. Dirac suggests that there are essentially three forms of Hamiltonian dynamics: The instant, the front and the point form. Unfortunately, one refers to the former as conventional 'quantization' (at equal usual time) and 'light-cone quantization' (at equal light-cone time). The three forms have a different initialization of the particle trajectories or the fields. They differ by the way one parametrizes four-dimensional space-time as illustrated in Fig. 1.

1 Four-vector conventions in the instant form

Lorentz vectors. Contra-variant four-vectors of position x^μ are written as

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z) = (x^0, \vec{x}_\perp, x^3) = (t, \vec{x}). \quad (1)$$

Covariant four-vectors x_μ are written as

$$x_\mu = (x_0, x_1, x_2, x_3) = (t, -x, -y, -z) = g_{\mu\nu} x^\nu. \quad (2)$$

The metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$ lower and raise the indices, respectively, and are given in Fig. 1. Implicit summation over repeated Lorentz (μ, ν, κ) or space

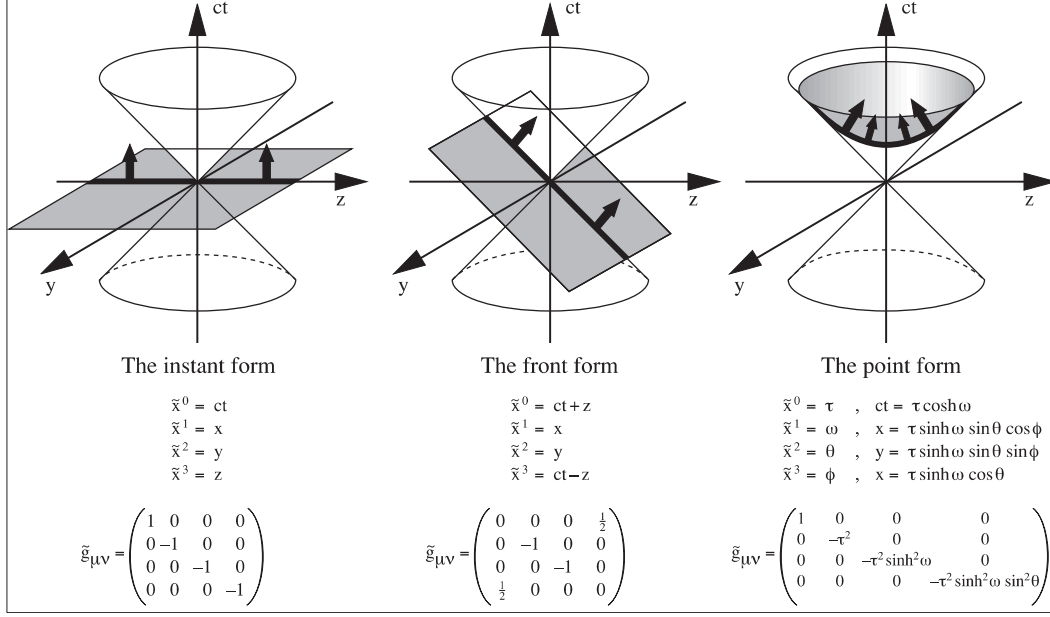


Fig. 1. Dirac's three forms of Hamiltonian dynamics.

(i, j, k) indices is understood. Scalar products are

$$x p = x^\mu p_\mu = x^0 p_0 + x^1 p_1 + x^2 p_2 + x^3 p_3 = tE - \vec{x} \cdot \vec{p}, \quad (3)$$

for example. The four-momentum of energy-momentum is $p^\mu = (p^0, p^1, p^2, p^3)$ or $p^\mu = (E, \vec{p})$.

Dirac matrices. The 4×4 Dirac matrices γ^μ are defined by

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \quad (4)$$

up to unitary transformations. The γ^k are anti-hermitean and γ^0 is hermitean. Useful combinations are $\alpha^k = \gamma^0 \gamma^k$ and $\beta = \gamma^0$, as well as

$$\sigma^{\mu\nu} = \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu), \quad \gamma_5 = \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3. \quad (5)$$

They usually are expressed in terms of the 2×2 Pauli matrices which are compiled here for completeness,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (6)$$

In Dirac representation the matrices are

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad (7)$$

$$\gamma_5 = \begin{pmatrix} 0 & +I \\ I & 0 \end{pmatrix}, \quad \alpha^k = \begin{pmatrix} 0 & \sigma^k \\ +\sigma^k & 0 \end{pmatrix}, \quad \sigma^{ij} = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \quad (8)$$

In chiral representation γ_0 and γ_5 are interchanged:

$$\gamma^0 = \begin{pmatrix} 0 & +I \\ I & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad (9)$$

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix}, \quad \sigma^{ij} = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \quad (10)$$

$(i, j, k) = 1, 2, 3$ are used cyclically.

Projection operators. Combinations of Dirac matrices like

$$\Lambda_+ = \frac{1}{2}(1 + \alpha^3) = \frac{\gamma^0}{2}(\gamma^0 + \gamma^3), \quad \Lambda_- = \frac{1}{2}(1 - \alpha^3) = \frac{\gamma^0}{2}(\gamma^0 - \gamma^3), \quad (11)$$

have often projector properties, particularly

$$\Lambda_+ + \Lambda_- = \mathbf{1}, \quad \Lambda_+ \Lambda_- = 0, \quad \Lambda_+^2 = \Lambda_+, \quad \Lambda_-^2 = \Lambda_-. \quad (12)$$

They are diagonal in the chiral and off-diagonal in the Dirac representation:

$$(\Lambda_+)_{\text{chiral}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\Lambda_+)_{\text{Dirac}} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \quad (13)$$

Dirac spinors. The solutions of the Dirac equations for mass m particles are the spinors $u(p, \lambda) \equiv u_\alpha(p, \lambda)$ and $v(p, \lambda) \equiv v_\alpha(p, \lambda)$,

$$(\not{p} - m) u(p, \lambda) = 0, \quad (\not{p} + m) v(p, \lambda) = 0. \quad (14)$$

They form an orthonormal set,

$$\bar{u}(p, \lambda) u(p, \lambda') = -\bar{v}(p, \lambda') v(p, \lambda) = 2m \delta_{\lambda\lambda'}, \quad (15)$$

which is complete,

$$\sum_{\lambda} u(p, \lambda) \bar{u}(p, \lambda) = \not{p} + m, \quad \sum_{\lambda} v(p, \lambda) \bar{v}(p, \lambda) = \not{p} - m. \quad (16)$$

Note that normalization here differs from most of the textbooks. The ‘Feynman slash’ is $\not{p} = p_{\mu} \gamma^{\mu}$. The Gordon decomposition of currents

$$\bar{u}(p, \lambda) \gamma^{\mu} u(q, \lambda') = \frac{1}{2m} \bar{u}(p, \lambda) \left((p + q)^{\mu} + i \sigma^{\mu\nu} (p - q)_{\nu} \right) u(q, \lambda') \quad (17)$$

is often useful. The relations

$$\gamma^{\mu} \not{a} \gamma_{\mu} = -2a, \quad \gamma^{\mu} \not{a} \not{b} \gamma_{\mu} = 4ab, \quad \gamma^{\mu} \not{a} \not{b} \not{c} \gamma_{\mu} = \not{c} \not{b} \not{a} \quad (18)$$

hold identically. With $\lambda = \pm 1$, the spin projection is $s = \lambda/2$.

Polarization vectors. The two polarization four-vectors $\epsilon_{\mu}(p, \lambda)$ are labeled by the spin projections $\lambda = \pm 1$. As solutions of the free Maxwell equations they are orthonormal and complete:

$$\epsilon^{\mu}(p, \lambda) \epsilon_{\mu}^{*}(p, \lambda') = -\delta_{\lambda\lambda'}, \quad p^{\mu} \epsilon_{\mu}(p, \lambda) = 0. \quad (19)$$

The star (*) refers to complex conjugation. The polarization sum is

$$d_{\mu\nu}(p) = \sum_{\lambda} \epsilon_{\mu}(p, \lambda) \epsilon_{\nu}^{*}(p, \lambda) = -g_{\mu\nu} + \frac{\eta_{\mu} p_{\nu} + \eta_{\nu} p_{\mu}}{p^{\kappa} \eta_{\kappa}}, \quad (20)$$

with the null vector $\eta^{\mu} \eta_{\mu} = 0$ given below.

2 Additional front-form conventions (LB)[3]

Lorentz vectors. Contra-variant four-vectors of position x^{μ} are written as

$$x^{\mu} = (x^{+}, x^1, x^2, x^{-}) = (x^{+}, \vec{x}_{\perp}, x^{-}). \quad (21)$$

Its time-like and space-like components are related to the instant form by

$$x^{+} = x^0 + x^3 \quad \text{and} \quad x^{-} = x^0 - x^3, \quad (22)$$

respectively, and referred to as ‘light-cone time’ and ‘light-cone space’. The covariant vectors are obtained by $x_\mu = g_{\mu\nu}x^\nu$, with the metric tensor(s)

$$g^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}. \quad (23)$$

Scalar products $xp = x^\mu p_\mu$ are

$$x^\mu p_\mu = x^+ p_- + x^- p_+ + x^1 p_1 + x^2 p_2 = \frac{1}{2}(x^+ p^- + x^- p^+) - \vec{x}_\perp \vec{p}_\perp. \quad (24)$$

All other four-vectors including γ^μ are treated correspondingly.

Dirac matrices. In Dirac representation of the γ -matrices holds

$$\gamma^+ \gamma^+ = \gamma^- \gamma^- = 0. \quad (25)$$

Alternating products are for example

$$\gamma^+ \gamma^- \gamma^+ = 4\gamma^+, \quad \gamma^- \gamma^+ \gamma^- = 4\gamma^-. \quad (26)$$

Projection operators. The projection matrices become

$$\Lambda_+ = \frac{1}{2}\gamma^0\gamma^+ = \frac{1}{4}\gamma^-\gamma^+, \quad \Lambda_- = \frac{1}{2}\gamma^0\gamma^- = \frac{1}{4}\gamma^+\gamma^-. \quad (27)$$

Dirac spinors. The Lepage-Brodsky representation is particularly simple:

$$u(p, \lambda) = \frac{1}{\sqrt{p^+}} (p^+ + \beta m + \vec{\alpha}_\perp \vec{p}_\perp) \times \begin{cases} \chi(\uparrow), & \text{for } \lambda = +1, \\ \chi(\downarrow), & \text{for } \lambda = -1, \end{cases} \quad (28)$$

$$v(p, \lambda) = \frac{1}{\sqrt{p^+}} (p^+ - \beta m + \vec{\alpha}_\perp \vec{p}_\perp) \times \begin{cases} \chi(\downarrow), & \text{for } \lambda = +1, \\ \chi(\uparrow), & \text{for } \lambda = -1. \end{cases} \quad (29)$$

The two χ -spinors are

$$\chi(\uparrow) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi(\downarrow) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}. \quad (30)$$

Table 1

Matrix elements of Dirac spinors $\bar{u}(p)\mathcal{M}u(q)$.

\mathcal{M}	$\frac{\bar{u}(p)\mathcal{M}u(q)}{\sqrt{p^+q^+}}\delta_{\lambda_p,\lambda_q}$	$\frac{\bar{u}(p)\mathcal{M}u(q)}{\sqrt{p^+q^+}}\delta_{\lambda_p,-\lambda_q}$
γ^+	2	0
γ^-	$\frac{2}{p^+q^+}(\vec{p}_\perp \cdot \vec{q}_\perp + m^2 + i\lambda_q \vec{p}_\perp \wedge \vec{q}_\perp)$	$\frac{2m}{p^+q^+}(p_\perp(\lambda_q) - q_\perp(\lambda_q))$
$\vec{\gamma}_\perp \cdot \vec{a}_\perp$	$\vec{a}_\perp \cdot \left(\frac{\vec{p}_\perp}{p^+} + \frac{\vec{q}_\perp}{q^+}\right) - i\lambda_q \vec{a}_\perp \wedge \left(\frac{\vec{p}_\perp}{p^+} - \frac{\vec{q}_\perp}{q^+}\right)$	$-a_\perp(\lambda_q) \left(\frac{m}{p^+} - \frac{m}{q^+}\right)$
1	$\frac{m}{p^+} + \frac{m}{q^+}$	$\frac{p_\perp(\lambda_q)}{q^+} - \frac{q_\perp(\lambda_q)}{p^+}$
$\gamma^- \gamma^+ \gamma^-$	$\frac{8}{p^+q^+}(\vec{p}_\perp \cdot \vec{q}_\perp + m^2 + i\lambda_q \vec{p}_\perp \wedge \vec{q}_\perp)$	$\frac{8m}{p^+q^+}(p_\perp(\lambda_q) - q_\perp(\lambda_q))$
$\gamma^- \gamma^+ \vec{\gamma}_\perp \cdot \vec{a}_\perp$	$\frac{4}{p^+}(\vec{a}_\perp \cdot \vec{p}_\perp - i\lambda_q \vec{a}_\perp \wedge \vec{p}_\perp)$	$-\frac{4m}{p^+}a_\perp(\lambda_q)$
$\vec{a}_\perp \cdot \vec{\gamma}_\perp \gamma^+ \gamma^-$	$\frac{4}{q^+}(\vec{a}_\perp \cdot \vec{q}_\perp + i\lambda_q \vec{a}_\perp \wedge \vec{q}_\perp)$	$\frac{4m}{q^+}a_\perp(\lambda_q)$
$\vec{a}_\perp \cdot \vec{\gamma}_\perp \gamma^+ \vec{\gamma}_\perp \cdot \vec{b}_\perp$	$2(\vec{a}_\perp \cdot \vec{b}_\perp + i\lambda_q \vec{a}_\perp \wedge \vec{b}_\perp)$	0
Notation: $\lambda = \pm 1, \quad a_\perp(\lambda) = -\lambda a_x - i a_y$ $\vec{a}_\perp \cdot \vec{b}_\perp = a_x b_x + a_y b_y, \quad \vec{a}_\perp \wedge \vec{b}_\perp = a_x b_y - a_y b_x.$ Symmetries: $\bar{v}(p) v(q) = -\bar{u}(q) u(p), \quad \bar{v}(p) \gamma^\mu v(q) = \bar{u}(q) \gamma^\mu u(p),$ $\bar{v}(p) \gamma^\mu \gamma^\nu \gamma^\rho v(q) = \bar{u}(q) \gamma^\rho \gamma^\nu \gamma^\mu u(p).$		

Tables of Dirac spinors. Matrix elements of Dirac spinors are given in Tables 1 and 2. As a particular application the Lorentz invariant spinor factor

$$S = j^\mu \bar{j}_\mu = [\bar{u}(k_1, \lambda_1) \gamma^\mu u(k'_1, \lambda'_1)] [\bar{v}(k'_2, \lambda'_2) \gamma_\mu v(k_2, \lambda_2)] \quad (31)$$

is calculated explicitly. It appears for example in the $q\bar{q}$ -scattering amplitude. In helicity space it can be understood as the matrix $\langle \lambda_1, \lambda_2 | S | \lambda'_1, \lambda'_2 \rangle$ whose matrix elements are functions of x, \vec{k}_\perp and x', \vec{k}'_\perp . For convenience, S is calculated here as $S = 2T \sqrt{x(1-x)x'(1-x')}$, *i.e.*

$$\langle \lambda_q, \lambda_{\bar{q}} | T | \lambda'_q, \lambda'_{\bar{q}} \rangle = \frac{1}{2} \frac{[\bar{u}(k_1, \lambda_1) \gamma^\mu u(k'_1, \lambda'_1)] [\bar{v}(k'_2, \lambda'_2) \gamma_\mu v(k_2, \lambda_2)]}{\sqrt{xx'} \sqrt{(1-x)(1-x')}}. \quad (32)$$

It is often useful to arrange S or T as a matrix in helicity space,

$$\langle \lambda_q, \lambda_{\bar{q}} | T | \lambda'_q, \lambda'_{\bar{q}} \rangle = \begin{matrix} & \uparrow\downarrow & \downarrow\uparrow & \uparrow\uparrow & \downarrow\downarrow \\ \begin{matrix} \uparrow\downarrow \\ \downarrow\uparrow \\ \uparrow\uparrow \\ \downarrow\downarrow \end{matrix} & \begin{pmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{pmatrix} \end{matrix}. \quad (33)$$

With $y \equiv 1 - x$ the diagonal elements are

$$T_{11} = \frac{m_1^2}{xx'} + \frac{m_2^2}{yy'} + \frac{\vec{k}_\perp^2}{xy} + \frac{\vec{k}'_\perp^2}{x'y'} + \frac{\vec{k}_\perp \cdot \vec{k}'_\perp + i\vec{k}_\perp \wedge \vec{k}'_\perp}{xx'} + \frac{\vec{k}_\perp \cdot \vec{k}'_\perp - i\vec{k}_\perp \wedge \vec{k}'_\perp}{yy'}, \quad (34)$$

$$T_{22} = T_{11}, \quad (35)$$

$$T_{33} = \frac{m_1^2}{xx'} + \frac{m_2^2}{yy'} + \frac{\vec{k}_\perp \cdot \vec{k}'_\perp + i\vec{k}_\perp \wedge \vec{k}'_\perp}{xyx'y'}, \quad (36)$$

$$T_{44} = T_{33}. \quad (37)$$

The off-diagonal matrix elements become

$$T_{12} = -m_1 m_2 \frac{(x - x')^2}{xyx'y'}, \quad T_{21} = -m_1 m_2 \frac{(x - x')^2}{xyx'y'}, \quad (38)$$

$$T_{13} = \frac{m_2}{yy'} x \left[\frac{k_\perp(\uparrow)}{x} - \frac{k'_\perp(\uparrow)}{x'} \right], \quad T_{31} = \frac{m_2}{yy'} x' \left[\frac{k_\perp(\downarrow)}{x} - \frac{k'_\perp(\downarrow)}{x'} \right], \quad (39)$$

$$T_{14} = \frac{m_1}{xx'} y \left[\frac{k_\perp(\downarrow)}{y} - \frac{k'_\perp(\downarrow)}{y'} \right], \quad T_{41} = \frac{m_1}{xx'} y' \left[\frac{k_\perp(\uparrow)}{y} - \frac{k'_\perp(\uparrow)}{y'} \right], \quad (40)$$

$$T_{23} = \frac{m_1}{xx'} y \left[\frac{k_\perp(\uparrow)}{y} - \frac{k'_\perp(\uparrow)}{y'} \right], \quad T_{32} = \frac{m_1}{xx'} y' \left[\frac{k_\perp(\downarrow)}{y} - \frac{k'_\perp(\downarrow)}{y'} \right], \quad (41)$$

$$T_{24} = \frac{m_2}{yy'} x \left[\frac{k_\perp(\downarrow)}{x} - \frac{k'_\perp(\downarrow)}{x'} \right], \quad T_{42} = \frac{m_2}{yy'} x' \left[\frac{k_\perp(\uparrow)}{x} - \frac{k'_\perp(\uparrow)}{x'} \right], \quad (42)$$

$$T_{34} = 0, \quad T_{43} = 0, \quad (43)$$

where $k_\perp(\uparrow) = -k_{\perp x} - ik_{\perp y}$ and $k_\perp(\downarrow) = k_{\perp x} - ik_{\perp y}$.

Close to kinematic equilibrium, *i.e.* for $x \sim x' = \bar{x}$ and $\vec{k}_\perp \sim \vec{k}'_\perp = 0$, the particles are essentially at rest, and S is diagonal ($\bar{x} = m_1/(m_1 + m_2)$):

$$\langle \lambda_1, \lambda_2 | S | \lambda'_1, \lambda'_2 \rangle = 4m_1 m_2 \delta_{\lambda_1, \lambda'_1} \delta_{\lambda_2, \lambda'_2}. \quad (44)$$

Far-off equilibrium S is also diagonal; all matrix elements vanish essentially as compared to the two leading ones (for $x \sim x'$ and $k'_\perp \gg k_\perp \gg m$):

$$\langle \uparrow\downarrow | S | \uparrow\downarrow \rangle = \langle \downarrow\uparrow | S | \downarrow\uparrow \rangle = 2\vec{k}'_\perp^2. \quad (45)$$

Table 2

Matrix elements of Dirac spinors $\bar{v}(p)\mathcal{M}u(q)$.

\mathcal{M}	$\frac{\bar{v}(p)\mathcal{M}u(q)}{\sqrt{p^+q^+}} \delta_{\lambda_p, \lambda_q}$	$\frac{\bar{v}(p)\mathcal{M}u(q)}{\sqrt{p^+q^+}} \delta_{\lambda_p, -\lambda_q}$
γ^+	0	2
γ^-	$\frac{2m}{p^+q^+} (p_\perp(\lambda_q) + q_\perp(\lambda_q))$	$\frac{2}{p^+q^+} (\vec{p}_\perp \cdot \vec{q}_\perp - m^2 + i\lambda_q \vec{p}_\perp \wedge \vec{q}_\perp)$
$\vec{\gamma}_\perp \cdot \vec{a}_\perp$	$a_\perp(\lambda_q) \left(\frac{m}{p^+} + \frac{m}{q^+} \right)$	$\vec{a}_\perp \cdot \left(\frac{\vec{p}_\perp}{p^+} + \frac{\vec{q}_\perp}{q^+} \right) - i\lambda_q \vec{a}_\perp \wedge \left(\frac{\vec{p}_\perp}{p^+} - \frac{\vec{q}_\perp}{q^+} \right)$
1	$\frac{p_\perp(\lambda_q)}{p^+} + \frac{q_\perp(\lambda_q)}{q^+}$	$-\frac{m}{p^+} + \frac{m}{q^+}$
$\gamma^- \gamma^+ \gamma^-$	$\frac{8m}{p^+q^+} (p_\perp(\lambda_q) + q_\perp(\lambda_q))$	$\frac{8}{p^+q^+} (\vec{p}_\perp \cdot \vec{q}_\perp - m^2 + i\lambda_q \vec{p}_\perp \wedge \vec{q}_\perp)$
$\gamma^- \gamma^+ \vec{\gamma}_\perp \cdot \vec{a}_\perp$	$\frac{4m}{p^+} a_\perp(\lambda_q)$	$\frac{4}{p^+} (\vec{a}_\perp \cdot \vec{p}_\perp - i\lambda_q \vec{a}_\perp \wedge \vec{p}_\perp)$
$\vec{a}_\perp \cdot \vec{\gamma}_\perp \gamma^+ \gamma^-$	$\frac{4m}{q^+} a_\perp(\lambda_q)$	$\frac{4}{q^+} (\vec{a}_\perp \cdot \vec{q}_\perp + i\lambda_q \vec{a}_\perp \wedge \vec{q}_\perp)$
$\vec{a}_\perp \cdot \vec{\gamma}_\perp \gamma^+ \vec{\gamma}_\perp \cdot \vec{b}_\perp$	0	$2(\vec{a}_\perp \cdot \vec{b}_\perp + i\lambda_q \vec{a}_\perp \wedge \vec{b}_\perp)$
Notation: $\lambda = \pm 1, \quad a_\perp(\lambda) = -\lambda a_x - i a_y$ $\vec{a}_\perp \cdot \vec{b}_\perp = a_x b_x + a_y b_y, \quad \vec{a}_\perp \wedge \vec{b}_\perp = a_x b_y - a_y b_x.$ Symmetries: $\bar{v}(p) v(q) = -\bar{u}(q) u(p), \quad \bar{v}(p) \gamma^\mu v(q) = \bar{u}(q) \gamma^\mu u(p),$ $\bar{v}(p) \gamma^\mu \gamma^\nu \gamma^\rho v(q) = \bar{u}(q) \gamma^\rho \gamma^\nu \gamma^\mu u(p).$		

Polarization vectors. The null vector is

$$\eta^\mu = (0, \vec{0}, 2). \quad (46)$$

The transversal polarization vectors $\vec{\epsilon}_\perp(\uparrow) = -\frac{1}{\sqrt{2}}(1, i)$ and $\vec{\epsilon}_\perp(\downarrow) = \frac{1}{\sqrt{2}}(1, -i)$ are the same as in Björken-Drell convention: Circular polarization has the spin projections $\lambda = \pm 1 = \uparrow\downarrow$. With the unit vectors \vec{e}_x and \vec{e}_y in p_x - and p_y -direction, respectively, one writes collectively

$$\vec{\epsilon}_\perp(\lambda) = \frac{-1}{\sqrt{2}}(\lambda \vec{e}_x + i \vec{e}_y). \quad (47)$$

The light-cone gauge $A^+ = 0$ induces $\epsilon^+(p, \lambda) = 0$, thus

$$\epsilon^\mu(p, \lambda) = \left(0, \vec{\epsilon}_\perp(\lambda), \frac{2\vec{\epsilon}_\perp(\lambda) \cdot \vec{p}_\perp}{p^+} \right), \quad (48)$$

which satisfies the transversality condition $p_\mu \epsilon^\mu(p, \lambda) = 0$ identically.

3 Canonical field theory for Quantum Chromodynamics

If one replaces each local gauge field $A^\mu(x)$ in QED by the matrix $\mathbf{A}^\mu(x)$,

$$A^\mu \longrightarrow (\mathbf{A}^\mu)_{cc'} = \frac{1}{2} \begin{pmatrix} \frac{1}{\sqrt{3}}A_8^\mu + A_3^\mu & A_1^\mu - iA_2^\mu & A_4^\mu - iA_5^\mu \\ A_1^\mu + iA_2^\mu & \frac{1}{\sqrt{3}}A_8^\mu - A_3^\mu & A_6^\mu - iA_7^\mu \\ A_4^\mu + iA_5^\mu & A_6^\mu + iA_7^\mu & -\frac{2}{\sqrt{3}}A_8^\mu \end{pmatrix}, \quad (49)$$

one generates Quantum Chromodynamics with its eight real valued color vector potentials A_a^μ , enumerated by the glue index $a = 1, 2, \dots, 8$. These matrices are all hermitean and traceless since the trace can always be absorbed into an Abelian U(1) gauge theory. Unitary transformations of this matrix belong to the class of special unitary 3×3 matrices, SU(3). In order to make sense of expressions like $\bar{\Psi} \mathbf{A}^\mu \Psi$ the quark fields $\Psi(x)$ must carry a color index $c = 1, 2, 3$ which is usually suppressed in the color triplet spinor $\Psi_{c,\alpha}(x)$.

More generally for SU(N), the vector potentials \mathbf{A}^μ are hermitian and traceless $N \times N$ matrices. All such matrices can be parametrized $\mathbf{A}^\mu \equiv T_{cc'}^a A_a^\mu$. The color index c (or c') runs now from 1 to n_c , and correspondingly the gluon index a (or r, s, t) from 1 to $n_c^2 - 1$. Both are implicitly summed over, with no distinction of lowering or raising them. The color matrices $T_{cc'}^a$ obey

$$[T^r, T^s]_{cc'} = if^{rsa} T_{cc'}^a \quad \text{and} \quad \text{Tr} (T^r T^s) = \frac{1}{2} \delta_r^s. \quad (50)$$

The *structure constants* f^{rst} for SU(3) are tabulated in the literature. For SU(2) they are the totally antisymmetric tensor ϵ_{rst} , since $T^a = \frac{1}{2} \sigma^a$ with σ^a being the Pauli matrices. For SU(3), the T^a are related to the Gell-Mann matrices λ^a by $T^a = \frac{1}{2} \lambda^a$, see also Eq.(49). The gauge-invariant Lagrangian density for QCD or SU(N) is then

$$\mathcal{L} = -\frac{1}{2} \text{Tr}(\mathbf{F}^{\mu\nu} \mathbf{F}_{\mu\nu}) + \frac{1}{2} [\bar{\Psi}(i\gamma^\mu \mathbf{D}_\mu - \mathbf{m})\Psi + \text{h.c.}]. \quad (51)$$

The factor $\frac{1}{2}$ is because of the trace convention in Eq.(50). The mass matrix $\mathbf{m} = m\delta_{cc'}$ is diagonal in color space. The matrix notation is particularly suited for establishing gauge invariance, with the unitary operators \mathbf{U} now being $N \times N$ matrices, hence *non-Abelian* gauge theory. Gauge invariance generates an extra term in the color-electro-magnetic fields

$$\mathbf{F}^{\mu\nu} \equiv \partial^\mu \mathbf{A}^\nu - \partial^\nu \mathbf{A}^\mu + ig[\mathbf{A}^\mu, \mathbf{A}^\nu]. \quad (52)$$

The covariant derivative *matrix* finally is $\mathbf{D}_{cc'}^\mu = \delta_{cc'}\partial^\mu + ig\mathbf{A}_{cc'}^\mu$, and the canonical energy-momentum stress tensor becomes

$$T^{\mu\nu} = 2\text{Tr}(\mathbf{F}^{\mu\kappa}\mathbf{F}_\kappa{}^\nu) + \frac{1}{2}[i\bar{\Psi}\gamma^\mu\mathbf{D}^\nu\Psi + \text{h.c.}] - g^{\mu\nu}\mathcal{L}. \quad (53)$$

Integrating it over a space-like hyper-surface, with the surface elements $d\omega_\lambda$ and the (finite) volume Ω defined most conveniently in terms of the totally antisymmetric tensor $\epsilon_{\lambda\mu\nu\rho}$ ($\epsilon_{+12-} = 1$),

$$d\omega_\lambda = \frac{1}{3!}\epsilon_{\lambda\mu\nu\rho}dx^\mu dx^\nu dx^\rho \quad \text{and} \quad \Omega = \int d\omega_+ = \int dx^1 dx^2 dx^-, \quad (54)$$

respectively, one arrives at the canonical energy-momentum four-vector

$$P^\nu = \int_\Omega d\omega_+ \left(2\text{Tr}(\mathbf{F}^{+\kappa}\mathbf{F}_\kappa{}^\nu) - g^{+\nu}\mathcal{L} + \frac{1}{2}[i\bar{\Psi}\gamma^+\mathbf{D}^\nu\Psi + \text{h.c.}] \right). \quad (55)$$

Note that both $T^{\mu\nu}$ and P^ν are *manifestly gauge-invariant*, and that all of this holds for SU(N), in fact that it holds for $d+1$ dimensions.

Since P^ν is gauge-independent and a constant of motion, it can be evaluated in the light-cone gauge $A^+ = 0$ and at the fixed light-cone time $x^+ = 0$. As initial condition one uses the free fields. For the fermions they are

$$\tilde{\Psi}_{acf}(x) = \sum_\lambda \int \frac{dk^+ d^2k_\perp}{\sqrt{2k^+(2\pi)^3}} \left(b_q u_\alpha(k, \lambda) e^{-ikx} + d_q^\dagger v_\alpha(k, \lambda) e^{+ikx} \right). \quad (56)$$

Each fermion is specified by the five numbers $q = (k^+, k_{\perp x}, k_{\perp y}, \lambda, c, f)$, *i.e.* by the three momenta, the helicity, color, and flavor, respectively. The fermion fields $\tilde{\Psi}$ become operator valued by the anti-commutation relations

$$\{b_q, b_{q'}^\dagger\} = \{d_q, d_{q'}^\dagger\} = \delta(k^+ - k'^+) \delta^{(2)}(\vec{k}_\perp - \vec{k}'_\perp) \delta_\lambda^{\lambda'} \delta_c^{c'} \delta_f^{f'}. \quad (57)$$

For the gauge bosons, the free fields are

$$\tilde{A}_\mu^a(x) = \sum_\lambda \int \frac{dk^+ d^2k_\perp}{\sqrt{2k^+(2\pi)^3}} \left(a_q \epsilon_\mu(k, \lambda) e^{-ikx} + a_q^\dagger \epsilon_\mu^*(k, \lambda) e^{+ikx} \right). \quad (58)$$

Each boson is specified by the four numbers $q = (k^+, k_{\perp x}, k_{\perp y}, \lambda, a)$, *i.e.* by the three momenta, the helicity, and glue, respectively. The operator structure of the boson fields is determined by

$$[a_q, a_{q'}^\dagger] = \delta(k^+ - k'^+) \delta^{(2)}(\vec{k}_\perp - \vec{k}'_\perp) \delta_\lambda^{\lambda'} \delta_a^{a'}. \quad (59)$$

Now the space integrations can be performed straight forwardly. At the end one remains with the P^ν as operators in Fock-space.

4 Energy-momentum as Fock-space operators

The three space-like momentum operators P^+ and \vec{P}_\perp are diagonal,

$$P^+ = \sum_\lambda \int dk^+ d^2 k_\perp \left(k_q^+ b_q^\dagger b_q + k_q^+ d_q^\dagger d_q + k_q^+ a_q^\dagger a_q \right), \quad (60)$$

$$\vec{P}_\perp = \sum_\lambda \int dk^+ d^2 k_\perp \left((\vec{k}_\perp)_q b_q^\dagger b_q + (\vec{k}_\perp)_q d_q^\dagger d_q + (\vec{k}_\perp)_q a_q^\dagger a_q \right). \quad (61)$$

Here as well as below the summation over color, glue and flavor is suppressed in the notation. Their eigenvalue is the same for all Fock states, *i.e.*

$$P^+ = \sum_{i \in \nu} (k^+)_i, \quad \vec{P}_\perp = \sum_{i \in \nu} (\vec{k}_\perp)_i, \quad (62)$$

where i runs over all particles in a Fock state ν . Since k^+ and thus P^+ are positive, one can introduce *longitudinal momentum fractions* $x_i = k_i^+/P^+$, and, due to boost-invariance, *intrinsic transversal momenta* $\vec{k}_{\perp i}$. They obey

$$\sum_{i \in \nu} (x)_i = 1, \quad \sum_{i \in \nu} (\vec{k}_\perp)_i = 0, \quad (63)$$

and will be used extensively below. Occasionally, the more compact notation

$$\int [d^3 q] \equiv \int dx_q \int d^2 k_{\perp q} \sum_{\lambda_q} \sum_{f_q} \sum_{c_q} \quad (64)$$

will be used.

The time-like component of P^ν is the analogue to the energy or the Hamiltonian in the conventional instant form of Hamiltonian dynamics and is highly off-diagonal. Rather than P^- , however, one considers the operator of invariant mass squared,


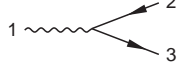
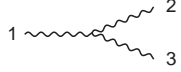
$$P_\nu P^\nu = P^+ P^- - \vec{P}_\perp^2 = P^+ P^- \equiv H_{\text{LC}}, \quad (65)$$

referred to as the ‘light-cone Hamiltonian H_{LC} [1]. This is reasonable, since P^- is multiplied only with the c-number P^+ . Unlike in the instant form, the

Table 3

The vertex interaction in terms of Dirac spinors. The matrix elements V_n are displayed on the right, the corresponding (energy) graphs on the left. Abbreviation:

$$\Delta_V = P^+ \frac{g}{\sqrt{16\pi^3}} \delta(k_1^+ - k_2^+ - k_3^+) \delta^{(2)}(\vec{k}_{\perp,1} - \vec{k}_{\perp,2} - \vec{k}_{\perp,3}).$$

	$V_1 = + \frac{\Delta_V}{\sqrt{k_1^+ k_2^+ k_3^+}} (\bar{u}_1 \not{\epsilon}_3 T^{a_3} u_2)$
	$V_3 = + \frac{\Delta_V}{\sqrt{k_1^+ k_2^+ k_3^+}} (\bar{v}_2 \not{\epsilon}_1^* T^{a_1} u_3)$
	$V_4 = \frac{i C_{a_2 a_3}^{a_1} \Delta_V}{\sqrt{k_1^+ k_2^+ k_3^+}} (\epsilon_1^* k_3) (\epsilon_2 \epsilon_3) \\ + \frac{i C_{a_2 a_3}^{a_1} \Delta_V}{\sqrt{k_1^+ k_2^+ k_3^+}} (\epsilon_3 k_1) (\epsilon_1^* \epsilon_2) \\ + \frac{i C_{a_2 a_3}^{a_1} \Delta_V}{\sqrt{k_1^+ k_2^+ k_3^+}} (\epsilon_3 k_2) (\epsilon_1^* \epsilon_2)$

front form Hamiltonian is additive in the free part T and the interaction U ,

$$H_{LC} = T + U. \quad (66)$$

The kinetic energy T is defined as that part of H_{LC} which is independent of the coupling constant and which can be interpreted as the free invariant mass-squared of the system. It is the sum of the three diagonal operators

$$T = \int [d^3 q] \left(\left(\frac{m^2 + \vec{k}_\perp^2}{x} \right)_q b_q^\dagger b_q + \left(\frac{m^2 + \vec{k}_\perp^2}{x} \right)_q d_q^\dagger d_q + \left(\frac{\vec{k}_\perp^2}{x} \right)_q a_q^\dagger a_q \right). \quad (67)$$

The interaction energy U breaks up into 20 different operators, grouped into

$$U = V + F + S. \quad (68)$$

The vertex interaction V is a sum of 4 operators

$$V = \int [d^3 q_1] \int [d^3 q_2] \int [d^3 q_3] \\ \left([b_1^\dagger b_2 a_3 V_1(1; 2, 3) + \text{h.c.}] + [d_1^\dagger d_2 a_3 V_2(1; 2, 3) + \text{h.c.}] \right. \\ \left. + [a_1^\dagger d_2 b_3 V_3(1; 2, 3) + \text{h.c.}] + [a_1^\dagger a_2 a_3 V_4(1; 2, 3) + \text{h.c.}] \right). \quad (69)$$

Table 4

The fork interaction in terms of Dirac spinors. The matrix elements $F_{n,j}$ are displayed on the right, the corresponding (energy) graphs on the left. Abbreviation:

$$\Delta = \frac{P^+}{2} \frac{g^2}{16\pi^3} \delta(k_1^+ + k_2^+ - k_3^+ - k_4^+) \delta^{(2)}(\vec{k}_{\perp,1} + \vec{k}_{\perp,2} - \vec{k}_{\perp,3} - \vec{k}_{\perp,4}).$$

	$F_1 = + \frac{2\Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \frac{(\bar{u}_1 T^a \gamma^+ u_2) (\bar{v}_3 \gamma^+ T^a u_4)}{(k_1^+ - k_2^+)^2}$
	$F_{3,1} = + \frac{\Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \frac{(\bar{u}_1 T^{a4} \not{\epsilon}_4 \gamma^+ \not{\epsilon}_3 T^{a2} u_2)}{(k_1^+ - k_4^+)^2}$
	$F_{3,2} = - \frac{2k_3^+ \Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \frac{(\bar{u}_1 T^a \gamma^+ u_2) (\epsilon_3 i C^a \epsilon_4)}{(k_1^+ - k_2^+)^2}$
	$F_{5,1} = + \frac{\Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \frac{(\bar{v}_3 T^{a1} \not{\epsilon}_1^* \gamma^+ \not{\epsilon}_2 T^{a2} u_4)}{(k_1^+ - k_3^+)^2}$
	$F_{5,2} = - \frac{\Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \frac{(\bar{v}_3 T^{a2} \not{\epsilon}_2 \gamma^+ \not{\epsilon}_1^* T^{a1} u_4)}{(k_1^+ - k_4^+)^2}$
	$F_{5,3} = + \frac{2(k_1^+ + k_2^+) \Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \frac{(\bar{v}_3 T^a \gamma^+ u_4) (\epsilon_1^* i C^a \epsilon_2)}{(k_1^+ - k_2^+)^2}$
	$F_{6,1} = + \frac{2k_3^+ (k_1^+ + k_2^+) \Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \frac{(\epsilon_1^* C^a \epsilon_2) (\epsilon_3 C^a \epsilon_4)}{(k_1^+ - k_2^+)^2}$
	$F_{6,2} = + \frac{2\Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} (\epsilon_1^* \epsilon_3) (\epsilon_2 \epsilon_4) C_{a_1 a_2}^a C_{a_3 a_4}^a$

It changes the particle number by 1. The *matrix elements* $V_n(1; 2, 3)$ are complex *c*-numbers with $V_2(1; 2, 3) = -V_1^*(1; 2, 3)$. They are functions of the various single-particle momenta k^+ , \vec{k}_\perp , helicities, colors and flavors and are tabulated in Table 3. As compact notation $b_i = b_{q_i}$ and $V_n(1; 2, 3) = V_n(q_1; q_2, q_3)$ is used. It should be emphasized that the graphs in these tables are *energy graphs* but *not Feynman diagrams*. They symbolize *matrix elements* but *not scattering amplitudes*. They conserve three-momentum but *not four-momentum*.

The fork interaction F is a sum of 6 operators,

$$\begin{aligned}
F = & \int [d^3 q_1] \int [d^3 q_2] \int [d^3 q_3] \int [d^3 q_4] \\
& \left(\left[b_1^\dagger b_2 d_3 b_4 F_1(1; 2, 3, 4) + \text{h.c.} \right] + \left[d_1^\dagger d_2 b_3 d_4 F_2(1; 2, 3, 4) + \text{h.c.} \right] \right. \\
& + \left[b_1^\dagger b_2 a_3 a_4 F_3(1; 2, 3, 4) + \text{h.c.} \right] + \left[d_1^\dagger d_2 a_3 a_4 F_4(1; 2, 3, 4) + \text{h.c.} \right] \\
& \left. + \left[a_1^\dagger a_2 d_3 b_4 F_5(1; 2, 3, 4) + \text{h.c.} \right] + \left[a_1^\dagger a_2 a_3 a_4 F_6(1; 2, 3, 4) + \text{h.c.} \right] \right). \quad (70)
\end{aligned}$$

It changes the particle number by 2. The matrix elements $F_n(1; 2, 3, 4)$ and their graphs are tabulated in Table 4, with $F_2 = F_1$ and $F_4 = F_3$.

Table 5

The seagull interaction in terms of Dirac spinors. The matrix elements $S_{n,j}$ are displayed on the right, the corresponding (energy) graphs on the left. Abbreviation:

$$\Delta = \frac{P^+}{2} \frac{g^2}{16\pi^3} \delta(k_1^+ + k_2^+ - k_3^+ - k_4^+) \delta^{(2)}(\vec{k}_{\perp,1} + \vec{k}_{\perp,2} - \vec{k}_{\perp,3} - \vec{k}_{\perp,4}).$$

	$S_1 = -\frac{\Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \frac{(\bar{u}_1 T^a \gamma^+ u_3) (\bar{u}_2 \gamma^+ T^a u_4)}{(k_1^+ - k_3^+)^2}$
	$S_{3,1} = +\frac{2\Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \frac{(\bar{u}_1 T^a \gamma^+ u_3) (\bar{v}_2 \gamma^+ T^a v_4)}{(k_1^+ - k_3^+)^2}$
	$S_{3,2} = -\frac{2\Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \frac{(\bar{v}_2 T^a \gamma^+ u_1) (\bar{v}_4 \gamma^+ T^a u_3)}{(k_1^+ + k_2^+)^2}$
	$S_{4,1} = +\frac{\Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \frac{(\bar{u}_1 T^{a4} \not{\epsilon}_4 \gamma^+ \not{\epsilon}_2^* T^{a2} u_3)}{(k_1^+ - k_4^+)$
	$S_{4,2} = +\frac{\Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \frac{(\bar{u}_1 T^{a2} \not{\epsilon}_2^* \gamma^+ \not{\epsilon}_4 T^{a4} u_3)}{(k_1^+ + k_2^+)$
	$S_{4,3} = +\frac{2(k_2^+ + k_4^+) \Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \frac{(\bar{u}_1 T^a \gamma^+ u_3) (\epsilon_2^* i C^a \epsilon_4)}{(k_1^+ - k_3^+)^2}$
	$S_{6,1} = +\frac{\Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \frac{(\bar{u}_1 T^{a3} \not{\epsilon}_3 \gamma^+ \not{\epsilon}_4 T^{a4} v_2)}{(k_1^+ - k_3^+)$
	$S_{6,2} = -\frac{(k_3^+ - k_4^+) \Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \frac{(\bar{u}_1 T^a \gamma^+ v_2) (\epsilon_3 i C^a \epsilon_4)}{(k_1^+ + k_2^+)^2}$
	$S_{7,1} = -\frac{(k_1^+ + k_3^+)(k_2^+ + k_4^+) \Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \frac{(\epsilon_1^* C^a \epsilon_3) (\epsilon_2^* C^a \epsilon_4)}{(k_1^+ - k_3^+)^2}$
	$S_{7,2} = +\frac{2k_3^+ k_4^+ \Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \frac{(\epsilon_1^* C^a \epsilon_2^*) (\epsilon_3 C^a \epsilon_4)}{(k_1^+ + k_2^+)^2}$
	$S_{7,3} = +\frac{\Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} (\epsilon_1^* \epsilon_3) (\epsilon_2^* \epsilon_4) C_{a_1 a_2}^a C_{a_3 a_4}^a$
	$S_{7,4} = +\frac{\Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} (\epsilon_1^* \epsilon_3) (\epsilon_2^* \epsilon_4) C_{a_1 a_4}^a C_{a_3 a_2}^a$
	$S_{7,5} = +\frac{\Delta}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} (\epsilon_1^* \epsilon_2^*) (\epsilon_3 \epsilon_4) C_{a_1 a_3}^a C_{a_2 a_4}^a$

The seagull interaction S is a sum of 7 operators

$$\begin{aligned}
S = & \int [d^3 q_1] \int [d^3 q_2] \int [d^3 q_3] \int [d^3 q_4] \\
& \left(b_1^\dagger b_2^\dagger b_3 b_4 S_1(1, 2; 3, 4) + d_1^\dagger d_2^\dagger d_3 d_4 S_2(1, 2; 3, 4) + b_1^\dagger d_2^\dagger b_3 d_4 S_3(1, 2; 3, 4) \right. \\
& \left. + b_1^\dagger a_2^\dagger b_3 a_4 S_4(1, 2; 3, 4) + d_1^\dagger a_2^\dagger d_3 a_4 S_5(1, 2; 3, 4) + \right.
\end{aligned}$$

$$+ [b_1^\dagger d_2^\dagger a_3 a_4 S_6(1, 2; 3, 4) + \text{h.c.}] + a_1^\dagger a_2^\dagger a_3 a_4 S_7(1, 2; 3, 4) \Big). \quad (71)$$

It does not change particle number. The matrix elements $S_n(1, 2; 3, 4)$ and their energy graphs are tabulated in Table 5, with $S_2 = S_1$ and $S_5 = S_4$.

Summarizing these considerations, one can state that the light-cone Hamiltonian $H \equiv H_{\text{LC}}$ consists of 23 operators with different operator structure.

5 The Sawicki transformation

When dealing with practical matters in the front form, particularly when working numerically, one is often pondered by the fact that the kinematical variables have a different range: $0 \leq x \leq 1$ and $-\infty \leq \vec{k}_\perp \leq \infty$. An elegant escape is to use a variable transform from x to k_z whose precursor had been introduced first by Sawicki [4].

The Sawicki transformation is demonstrated here at hand of the example:

$$M^2 \psi(x, \vec{k}_\perp) = \left[\frac{m_1^2 + \vec{k}_\perp^2}{x} + \frac{m_2^2 + \vec{k}_\perp^2}{1-x} \right] \psi(x, \vec{k}_\perp) - \frac{m_1 m_2}{\pi^2} \int_0^1 dx' \int_{-\infty}^{\infty} d^2 \vec{k}'_\perp \frac{1}{\sqrt{x(1-x)x'(1-x')}} \frac{\alpha}{Q^2} \psi(x', \vec{k}'_\perp). \quad (72)$$

The physical interpretation of this integral equation in the three variables x and \vec{k}_\perp is given below. M^2 is the invariant-mass squared eigenvalue and $\psi(x, \vec{k}_\perp)$ is the associated eigenfunction. It is the probability amplitude for finding a particle of mass m_1 with momentum fraction x and transversal momentum \vec{k}_\perp , and correspondingly an anti-particle of mass m_2 with $1-x$ and $-\vec{k}_\perp$. The inverse of the kernel is defined as

$$Q^2(x, \vec{k}_\perp, x', \vec{k}'_\perp) = -\frac{1}{2} \left[(k_1 - k'_1)^2 + (k_2 - k'_2)^2 \right], \quad (73)$$

thus

$$Q^2 = (\vec{k}_\perp - \vec{k}'_\perp)^2 - \frac{1}{2} (x - x') \left(\frac{m_1^2 + k_\perp^2}{x} - \frac{m_1^2 + k'^2_\perp}{x'} \right) + \frac{1}{2} (x - x') \left(\frac{m_2^2 + k_\perp^2}{1-x} - \frac{m_2^2 + k'^2_\perp}{1-x'} \right). \quad (74)$$

The coupling constant is α .

Integration variables can be changed from x to k_z by introducing

$$x(k_z) = \frac{E_1 + k_z}{E_1 + E_2}, \text{ with } E_{1,2} = \sqrt{m^2 + k_z^2 + \vec{k}_\perp^2}. \quad (75)$$

The new integration variable k_z can be combined with \vec{k}_\perp into $\vec{k} = (\vec{k}_\perp, k_z)$ which however is not a 3-vector in the usual sense. The Jacobian is

$$\frac{dx}{x(1-x)} = \frac{1}{A(k)} \frac{dk_z}{m_r}, \quad (76)$$

with the dimensionless function

$$A(k) = \frac{1}{m_r} \frac{E_1 E_2}{E_1 + E_2}. \quad (77)$$

The reduced mass and the sum mass is

$$\frac{1}{m_r} = \frac{1}{m_1} + \frac{1}{m_2}, \quad m_s = m_1 + m_2. \quad (78)$$

The free invariant mass-squared on the r.h.s. of Eq.(72), *i.e.*

$$\frac{m_1^2 + \vec{k}_\perp^2}{x} + \frac{m_2^2 + \vec{k}_\perp^2}{1-x}, \quad (79)$$

plays the role of a the kinetic energy T . Rewriting it, one uses first

$$\frac{m_1^2 + \vec{k}_\perp^2}{x} = \frac{m_1^2 + \vec{k}_\perp^2}{E_1 + k_z} (E_1 + E_2) = (E_1 - k_z) (E_1 + E_2), \quad (80)$$

and correspondingly

$$\frac{m_2^2 + \vec{k}_\perp^2}{1-x} = (E_2 + k_z) (E_1 + E_2). \quad (81)$$

The kinetic energy can thus be rewritten identically as

$$T(k) = \frac{m_1^2 + \vec{k}_\perp^2}{x} + \frac{m_2^2 + \vec{k}_\perp^2}{1-x} - m_s^2 \equiv C(k) \vec{k}^2, \quad (82)$$

with the dimensionless function

$$C(k) = (E_1 + m_1 + E_2 + m_2) \left(\frac{1}{E_1 + m_1} + \frac{1}{E_2 + m_2} \right). \quad (83)$$

If the wave function is substituted according to

$$\psi(x, \vec{k}_\perp) = \phi(x, \vec{k}_\perp) \sqrt{\frac{A(x, \vec{k}_\perp)}{x(1-x)}}, \quad (84)$$

one converts the integral equation (72) into an integral equation

$$\left[M^2 - m_s^2 - C(k) \vec{k}^2 \right] \phi(\vec{k}) = -\frac{m_s}{\pi^2} \int \frac{d^3 \vec{k}'}{\sqrt{A(k)A(k')}} \frac{\alpha}{Q^2(\vec{k}, \vec{k}')} \phi(\vec{k}'), \quad (85)$$

like in usual momentum space. However, since no changes have been made except relabeling the integration variable, it is identical with Eq.(72), and thus is a genuine front-form or light-cone integral equation.

The above simplifies considerably for equal particle masses $m_1 = m_2 = m$:

$$m_r = \frac{m}{2}, \quad m_s = 2m, \quad (86)$$

and

$$x(k_z) = \frac{1}{2} \left(1 - \frac{k_z}{\sqrt{m^2 + \vec{k}_\perp^2 + k_z^2}} \right), \quad k_z(x) = \left(x - \frac{1}{2} \right) \sqrt{\frac{m^2 + k_\perp^2}{x(1-x)}}. \quad (87)$$

The coefficient functions A and C become

$$A(k) = \sqrt{1 + \frac{k^2}{m^2}}, \quad C(k) = 4. \quad (88)$$

In a non-relativistic situation holds $k^2 \ll m^2$, thus

$$A(k) \sim 1, \quad Q^2 \sim (\vec{k} - \vec{k}')^2. \quad (89)$$

To substitute that in the kernel of an integral equation like (85) is certainly not justified, since the integration variable has $k' \rightarrow \infty$ at the upper limit. But if one does it anyway, *i.e.*

$$\left[M^2 - 4m^2 - 4\vec{k}^2 \right] \phi(\vec{k}) = -\frac{2m}{\pi^2} \int d^3 \vec{k}' \frac{\alpha}{(\vec{k} - \vec{k}')^2} \phi(\vec{k}'), \quad (90)$$

one can generate analytical solutions, see Sect. 6. The ground state has

$$M^2 = m^2(4 - \alpha^2), \quad \phi(k) = \frac{1}{\left(1 + \frac{k^2}{p_B^2}\right)^2}, \quad (91)$$

with $p_B = m\alpha/2$. When substituting the inverse Sawicki transformation back into the defining Eq.(84),

$$\vec{k}^2 = \vec{k}_\perp^2 + k_z^2 = \frac{m^2(2x-1)^2 + \vec{k}_\perp^2}{4x(1-x)}, \quad (92)$$

one expresses the (un-normalized) light-cone wavefunction in terms of the light-cone variables x and \vec{k}_\perp ,

$$\begin{aligned} \psi(x, \vec{k}_\perp) &= \frac{1}{\sqrt{x(1-x)}} \frac{1}{\left(1 + \frac{m^2(2x-1)^2 + \vec{k}_\perp^2}{\alpha^2 m^2 x(1-x)}\right)^2} \\ &\times \left(1 + \frac{m^2(2x-1)^2 + \vec{k}_\perp^2}{4m^2 x(1-x)}\right)^{\frac{1}{4}}. \end{aligned} \quad (93)$$

The last factor can also be set to unity because of $A \sim 1$. Even then, the present $\psi(x, \vec{k}_\perp)$ differs considerably from a similar expression in the literature [3]. In particular the factor $\sqrt{x(1-x)}$ is different. Here is the part of $\psi(x, \vec{k}_\perp)$ which violates rotational invariance as it should!

6 Fourier transforms in the front form

The interpretation of front-form equations such as Eq.(72) in terms of physics is not always easy, even if they are Sawicki-transformed to a momentum space integral equation like (85). Our intuition and our language is shaped in configuration space. To boost that one can apply Fourier transformation, but only in principle, since an exact but analytical Fourier transformation of Eq.(85) for example is close to impossible. But one can apply simplifying assumptions.

The steps are demonstrated at hand of the example

$$\left[M^2 - m_s^2 - \frac{m_s}{m_r} \vec{k}^2\right] \phi(\vec{k}) = -\frac{m_s}{\pi^2} \int d^3 \vec{k}' \left(\frac{\alpha}{(\vec{k} - \vec{k}')^2} + \frac{\alpha}{2m_r m_s} \right) \phi(\vec{k}'). \quad (94)$$

Table 6

Compilation of a few familiar Fourier transforms.

$U(\vec{q})$	$F[U] = \int d^3\vec{q} \, e^{-i\vec{q}\vec{x}} U(\vec{q})$	$U(\vec{q})$	$F[U] = \int d^3\vec{q} \, e^{-i\vec{q}\vec{x}} U(\vec{q})$
$\frac{1}{q^2 + \mu^2}$	$\frac{2\pi^2}{r} e^{-\mu r}$	$\frac{1}{q^2}$	$\frac{2\pi^2}{r}$
$\frac{\mu^2}{(q^2 + \mu^2)^2}$	$\pi^2 \mu e^{-\mu r}$	$\frac{1}{\mu^2}$	$\frac{(2\pi)^3}{\mu^2} \delta^{(3)}(\vec{x})$

It is identical with Eq.(85), except for the simple additive constant $\alpha/(2m_r m_s)$ in the kernel. What is its interpretation?

Transforming the momentum-space wave function to one in configuration space by $\psi(\vec{x}) = (2\pi)^{-\frac{3}{2}} \int d^3\vec{p} \, e^{-i\vec{p}\vec{x}} \phi(\vec{p})$, and using the expressions compiled in Table 6, one gets

$$\left[M^2 - m_s^2 + \frac{m_s}{m_r} \vec{\nabla}^2 \right] \psi(\vec{r}) = 2m_s V(r) \psi(\vec{r}). \quad (95)$$

The function $V(r)$ has the dimension of a mass and is defined by

$$V(r) = -\frac{\alpha}{r} - 2\frac{\alpha\pi}{m_r m_s} \delta^{(3)}(\vec{r}). \quad (96)$$

The strange ‘2’ finds its explanation as the gyro-magnetic ratio for a fermion, by analogy with the hyperfine interaction in hydrogen in the singlet channel,

$$V_{\text{hf-s}}(r) = -\frac{\alpha}{r} - g_p \frac{\alpha\pi}{m_e m_p} \delta^{(3)}(\vec{r}). \quad (97)$$

Here, m_p and g_p are the protons mass and gyro-magnetic ratio, respectively and m_e is the mass of the electron. If the mass-square eigenvalue is substituted by an energy eigenvalue E , defined by

$$M^2 = m_s^2 + m_s E, \quad (98)$$

one gets the conventional Schrödinger equation for the Coulomb plus a contact potential,

$$-\frac{\nabla^2}{2m_r} \psi(\vec{r}) + V(r) \psi(\vec{r}) = E \psi(\vec{r}). \quad (99)$$

If the latter is suppressed, the analytical solution for the ground state is

$$\psi(\vec{r}) = e^{-p_B \vec{r}}, \quad E = \frac{p_B^2}{2m_r}, \quad (100)$$

with the Bohr momentum $p_B = m_r \alpha$. If one includes the contact term, one gets in first order perturbation theory the usual

$$E_{\text{perturbative}} = \frac{p_B^2}{2m_r} + \alpha \frac{2\pi}{m_r m_s} |\psi(0)|^2. \quad (101)$$

The size in either case is

$$\langle r^2 \rangle = \frac{3}{p_B^2} = \frac{3}{m_r^2 \alpha^2}. \quad (102)$$

But Eq.(96) has no solution in the proper sense! A Dirac-delta function is no proper function and must be regulated, for instance by a Yukawa potential

$$V(r) = -\frac{\alpha}{r} - \frac{\alpha \mu^2}{m_r m_s} \frac{e^{-\mu r}}{r}. \quad (103)$$

Transforming back to momentum space gives

$$\left[M^2 - m_s^2 - \frac{m_s}{m_r} \vec{k}^2 \right] \phi(\vec{k}) = -\frac{\alpha}{\pi^2} \int d^3 k' \left(\frac{m_s}{Q^2} + \frac{1}{2m_r} \frac{\mu^2}{\mu^2 + Q^2} \right) \phi(\vec{k}'). \quad (104)$$

Note that the coupling constant of the Yukawa potential $\beta = \alpha \mu^2 / (m_r m_s)$ is potentially very large, since one must go to the limit $\mu \rightarrow \infty$ to retrieve the Dirac-delta-function. It is not trivial to solve Eqs.(103) or (104).

References

- [1] S.J. Brodsky, H.C. Pauli and S.S. Pinsky, *Physics Reports* **301**, 299 (1998).
- [2] P.A.M. Dirac, *Rev. Mod. Phys.* **21**, 392 (1949).
- [3] G.P. Lepage and S.J. Brodsky, *Phys. Rev.* **D22**, 2157 (1980).
- [4] M.Sawicki, *Phys.Rev.* **D32**, 2666 (1985).